(APG- AL

Analytical Laboratory Report 62-AL-154
24 August 1962

Title: Computer Program for Analysis of Sensitivity Data Following a Normal Distribution

OMS No.: 4230.1.1041.00

Prepared for: Engineering Testing

AD661686

INTRODUCTION

The term "sensitivity test" is used in statistical literature to denote any experiment from which quantal data (yes-no, or success-failure response) are observed as the intensity of a stimulus is varied. The stimulus may be the striking velocity of a projectile and the response the penetration or non-penetration of armor; or the stimulus may be the distance from muzzle to target and the response the functioning or non-functioning of a fuze, and so on. It is customary in the analysis of such experiments to use the integrated normal distribution as the model for describing the probability of "success" as the intensity of the stimulus varies. (Other models such as the Logistics Curve are also used though less widely than the Normal Curve.)

Analysis of sensitivity data following the normal model requires considerable computation except in very special situations. To facilitate making such analyses the computation of the maximum likelihood estimates of the unknown parameters and related statistics was programmed for the Bendix G15-D Computer. The program is intended to be of assistance to engineering personnel of D&PS in their analyses of tests of armor plate, penetration tests of ammunition, and in tests of fuzes for arming distance. The program would normally not be used to calculate the ballistic limit for a small test, but could be used in the analysis of a series of small tests.

This report explains the use of the program, the input data, formulas used, and output data, and gives some comments on the applicability of this type of analysis to tests involving a small number of rounds. This work was carried out as part of project "Procedures and Instrumentation on Techniques for Industrial Testing".

THEORY

The theory of estimation by the method of maximum likelihood is developed in statistical texts such as Cramèr, Mathematical Methods of Statistics, and its application to sensitivity analysis is described in a number of publications, e.g. Dixon and Mood "A Method for Obtaining and Analyzing Sensitivity Data" Journal of the American Statistical Association, Vol. 43 (1948) and BRL Technical Note 151. A brief statement of some results of the theory, however,

14

must be given here so that data computed by the machine program can be properly interpreted.

Model

The probability (p) of success with stimulus x is assumed to be

$$P = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt$$

where μ and σ are unknown parameters.

Let Y_i be a random variable taking values 1 or 0 depending on whether the outcome of the trial is a success or failure at level x_i of the stimulus.

Let Prob $(Y_i=1) = p_i$. Then Prob $(Y_i=0) = 1 - p_i$.

Estimation of Parameters μ and σ

The likelihood equation is

$$L = \prod_{i=1}^{n} p_i^{\,y_i}(1 - p_i)^{1-y_i} \quad \text{where } p_i \text{ is a function of } \mu, \sigma.$$

Solution of equations

$$\frac{\partial \log L}{\partial \mu} = \frac{\partial \log L}{\partial \sigma} = 0$$

for μ and σ gives the maximum likelihood estimates $\hat{\mu}$ and $\hat{\sigma}$.

Since the two equations above are not directly soluble, an iteration scheme is employed. The equations for iteration, in matrix notation, are:

$$\begin{bmatrix} \dfrac{\partial^2 \log \mathcal{L}}{\partial \mu^2} & \dfrac{\partial^2 \log \mathcal{L}}{\partial \mu\,\partial \sigma} \\[2ex] \dfrac{\partial^2 \log \mathcal{L}}{\partial \mu\,\partial \sigma} & \dfrac{\partial^2 \log \mathcal{L}}{\partial \sigma^2} \end{bmatrix}_{\mu_0,\sigma_0} \begin{bmatrix} \Delta\mu \\[2ex] \Delta\sigma \end{bmatrix} = - \begin{bmatrix} \dfrac{\partial \log \mathcal{L}}{\partial \mu} \\[2ex] \dfrac{\partial \log \mathcal{L}}{\partial \sigma} \end{bmatrix}_{\mu_0,\sigma_0}$$

where μ_0, σ_0 is a trial value. Solution of these equations (linear) for $\Delta\mu$ and $\Delta\sigma$ gives the first provisional solution as

$$\mu_1 = \mu_0 + \Delta_1\mu$$

$$\sigma_1 = \sigma_0 + \Delta_1\sigma$$

The iteration equations are re-evaluated with μ_1, σ_1 replacing μ_0, σ_0 and a second provisional solution is found.

$$\mu_2 = \mu_1 + \Delta_2\mu$$

$$\sigma_2 = \sigma_1 + \Delta_2\sigma$$

The iteration is continued until $\Delta\mu$ and $\Delta\sigma$ are "small". The final provisional solution will be called the maximum likelihood solutions and denoted $\hat{\mu}, \hat{\sigma}$.

Estimation of Precision

Maximum likelihood theory shows that $\hat{\mu}$ and $\hat{\sigma}$ (for large samples) are jointly normally distributed. The theory also shows that any provisional solution has the same asymptotic distribution. Since the distributions are the same no loss in accuracy is incurred by treating the provisional solution as the maximum likelihood estimate. In large samples $(\hat{\mu}-\mu)$ and $(\hat{\sigma}-\sigma)$ are approximately normally distributed with zero means and covariance matrix, the inverse of A where

$$A = -\mathcal{E}\begin{bmatrix} \dfrac{\partial^2 \log \mathcal{L}}{\partial \mu^2} & \dfrac{\partial^2 \log \mathcal{L}}{\partial \mu\, \partial \sigma} \\ \dfrac{\partial^2 \log \mathcal{L}}{\partial \mu\, \partial \sigma} & \dfrac{\partial^2 \log \mathcal{L}}{\partial \sigma^2} \end{bmatrix}$$

PROGRAM

This problem has been handled as three separate programs corresponding to three different configurations of data.

I. Grouped Data. This class is characterized by more than one trial at each x_i, or by a large number of trials for which the x_i can be grouped for convenience according to common statistical practice.

II. Ungrouped Data. Here the number of trials is small and only one trial is conducted at each x_i. In practice more than one trial may have been performed at some values of x, but the trials are treated as though the x values

are distinct. This part is further subdivided because of a peculiarity of the method of estimation when no zone or mixed results (ZMR) occur. Call x_H the largest x_i for which a failure occurred, and x_L the smallest x_i for which a success occurred. A zone of mixed results exists if and only if $x_H > x_L$. The program for this situation is called "Ungrouped Data, ZMR."

When $x_H \leq x_L$ there is no zone of mixed results and the estimation procedure used in the foregoing fails because the estimate of the scale parameter (σ) is zero. This situation, though regarded as an exception, occurs quite often in small samples. The data provide no estimate of σ ; μ can, however, be estimated if σ is known or can be assigned a reasonable value, say no less than one-half nor more than twice the true value. This program is labelled "Ungrouped Data, No. ZMR."

The details of data input, formulas for machine use, and of typed out results are contained in the paragraphs that follow. Although each program is treated separately formulas for "Ungrouped Data, No. ZMR" are given as modifications of those in the program for "Ungrouped Data, ZMR".

In the formulas of all three programs, use is made of the following symbols not previously defined.

$t_i = (x_i - \mu)/\sigma$	where μ and σ may have affixes denoting them as provisional or maximum likelihood solutions.
$Z(t_i) = z_i = \dfrac{e^{-t_i^2/2}}{\sqrt{2\pi}}$	Ordinate of the normal density
$v(t_i) = \dfrac{z(t_i)}{1-p(t_i)}$	(See Inclosure 1.)
$d > 0$	A constant chosen for each set of data (in the units of x) to halt the iteration process. The process halts when $\lvert\Delta\mu\rvert + \lvert\Delta\sigma\rvert < d$.

<u>Grouped Data</u>

1. Data for input are:

   (a) A set of ordered values x_i, the number of trials at level x_i, and the number of successes at that level.

x_i = level of stimulus

n_i = number of trials at level x_i

m_i = number of successes at level x_i.

x_i	n_i	m_i
---	---	---
---	---	---
	etc.	

(b) A first guess (μ_o, σ_o) of the unknown parameters. These can usually be estimated from a plot of the data on normal probability paper.

(c) A value d.

NOTE: The program will take a maximum of 15 values of x. These need not be equally spaced.

2. $\hat{\mu}$ and $\hat{\sigma}$ are calculated by the following process:

(a) For each x_i calculate

$t_i = (x_i - \mu_o)/\sigma_o$

$v(t_i)$ and $v(-t_i)$

(b) Form the following sums

$a = \sum(n_i-m_i)\ v(t_i)$	$a' = \sum m_i\ v(-t_i)$
$b = \sum(n_i-m_i)\ t_i\ v(t_i)$	$b' = \sum m_i\ t_i\ v(-t_i)$
$c = \sum(n_i-m_i)\ [v(t_i)]^2$	$c' = \sum m_i\ [v(-t_i)]^2$
$d = \sum(n_i-m_i)\ t_i^2\ v(t_i)$	$d' = \sum m_i\ t_i^2\ v(-t_i)$
$e = \sum(n_i-m_i)\ t_i\ [v(t_i)]^2$	$e' = \sum m_i\ t_i\ [v(-t_i)]^2$
$f = \sum(n_i-m_i)\ t_i^3\ v(t_i)$	$f' = \sum m_i\ t_i^3\ v(-t_i)$
$g = \sum(n_i-m_i)\ t_i^2\ [v(t_i)]^2$	$g' = \sum m_i\ t_i^2\ [v(-t_i)]^2$

(c) Solve the following 2 equations for $\Delta\mu$ and $\Delta\sigma$.

$$-(a-a')/\sigma_0 = \frac{b-c-b'-c'}{\sigma_0^2}\Delta\mu + \frac{d-e-a-d'-e'+a'}{\sigma_0^2}\Delta\sigma$$

$$-(b-b')/\sigma_0 = \frac{d-e-a-d'-e'+a'}{\sigma_0^2}\Delta\mu + \frac{f-2b-g-f'+2b'-g'}{\sigma_0^2}\Delta\sigma$$

(d) Then $\mu_1 = \mu_0 + \Delta_1\mu$

$\sigma_1 = \sigma_0 + \Delta_1\sigma$

(e) Repeat steps (a), (b) and (c) with μ_0, σ_0 replaced by μ_1, σ_1. Continue the iteration until the stopping rule halts the iteration. The last values μ_k, σ_k are taken as $\hat{\mu}$, $\hat{\sigma}$.

3. Precision of $\hat{\mu}, \hat{\sigma}$ is calculated as follows:

(a) For each x_i calculate

$t_i = (x_i - \hat{\mu})/\hat{\sigma}$

z_i, $v(t_i)$ and $v(-t_i)$

(b) Form the following

$$A = \frac{\sum n_i z_i [v(t_i) + v(-t_i)]}{\hat{\sigma}^2}$$

$$B = \frac{\sum n_i t_i z_i [v(t_i) + v(-t_i)]}{\hat{\sigma}^2}$$

$$C = \frac{\sum n_i t_i^2 z_i [v(t_i) + v(-t_i)]}{\hat{\sigma}^2}$$

$AC - B^2 = D$

$\text{var } \hat{\mu} = C/D$

$\text{var } \hat{\sigma} = A/D$

$\text{cov } \hat{\mu}\hat{\sigma} = -B/D$

4. The following results are typed out by the computer.

   (a) μ_0, σ_0 and successive provisional solutions, as well as successive values $\Delta\mu$ and $\Delta\sigma$.

   (b) D, var $\hat{\mu}$, var $\hat{\sigma}$ and cov $\hat{\mu}\hat{\sigma}$.

<u>Ungrouped Data, ZMR</u>

1. Data for input are:

   (a) A set of unordered values x_i and an associated indicator of "success" or "failure" δ_i. $\delta = 1$ if success, $\delta = 0$ if failure.

x_i	δ_i
--	--
--	--

etc.

   (b) A first guess (μ_0, σ_0) of the unknown parameters (optional). If no special first guess is desired, the computer makes a first estimate from the data.

   (c) A value d.

2. $\hat{\mu}$ and $\hat{\sigma}$ are calculated by the following process:

   (a) Examine x_i to find x_H and x_L.

   $\mu_0 = (x_H + x_L)/2$

   $\sigma_0 = x_H - x_L$

   (b) For each x_i calculate

   $t_i = (x_i - \mu_0)/\sigma_0$

   $v(t_i)$ for x_i having $\delta = 0$

   $v(-t_i)$ for x_i having $\delta = 1$.

(c) Form the following sums

For x's having $\delta = 0$	For x's having $\delta = 1$
$a = \sum v(t_i)$	$a' = \sum v(-t_i)$
$b = \sum t_i\, v(t_i)$	$b' = \sum t_i\, v(-t_i)$
$c = \sum [v(t_i)]^2$	$c' = \sum [v(-t_i)]^2$
$d = \sum t_i^2\, v(t_i)$	$d' = \sum t_i^2\, v(-t_i)$
$e = \sum t_i [v(t_i)]^2$	$e' = \sum t_i [v(-t_i)]^2$
$f = \sum t_i^3\, v(t_i)$	$f' = \sum t_i^3\, v(-t_i)$
$g = \sum t_i^2 [v(t_i)]^2$	$g' = \sum t_i^2 [v(-t_i)]^2$

(d) Solve the following 2 equations for $\Delta\mu$ and $\Delta\sigma$.

$$-(a - a')/\sigma_0 = \frac{b - c - b' - c'}{\sigma_0^2}\,\Delta\mu + \frac{d - e - a - d' - e' + a'}{\sigma_0^2}\,\Delta\sigma$$

$$-(b - b')/\sigma_0 = \frac{d - e - a - d' - e' + a'}{\sigma_0^2}\,\Delta\mu + \frac{f - 2b - g - f' + 2b' - g'}{\sigma_0^2}\,\Delta\sigma$$

(e) Then $\mu_1 = \mu_0 + \Delta_1\mu$

$\sigma_1 = \sigma_0 + \Delta_1\sigma$

(f) Repeat steps (b), (c) and (d) with μ_0, σ_0 replaced by μ_1, σ_1. Continue the iteration until the stopping rule halts the iteration. The last values μ_k, σ_k are taken as $\hat{\mu}, \hat{\sigma}$.

3. Precision of $\hat{\mu}, \hat{\sigma}$ is calculated as follows: (optional)

(a) For each x_i calculate

$t_i = (x_i - \hat{\mu})/\hat{\sigma}$

z_i, $v(t_i)$ and $v(-t_i)$

(b) Form the following

$$A = \frac{\sum z_i \, [v(t_i) + v(-t_i)]}{\hat{\sigma}^2}$$

$$B = \frac{\sum t_i \, z_i \, [v(t_i) + v(-t_i)]}{\hat{\sigma}^2}$$

$$C = \frac{\sum t_i^2 \, z_i \, [v(t_i) + v(-t_i)]}{\hat{\sigma}^2}$$

$D = AC - B^2$

$\text{var}\, \hat{\mu} = C/D$

$\text{var}\, \hat{\sigma} = A/D$

$\text{cov}\, \hat{\mu}\hat{\sigma} = -B/D$

4. The following results are typed out by the computer

   (a) μ_0, σ, and successive provisional solutions, as well as successive values $\Delta\mu$ and $\Delta\sigma$.

   (b) D, var $\hat{\mu}$, var $\hat{\sigma}$ and cov $\hat{\mu}\hat{\sigma}$ (optional)

<u>Ungrouped Data, No ZMR</u> (Expressed as a modification of the preceding program)

1. Data for input are:

   (a) No change.

   (b) An assumed or known value of σ. μ is estimated by the computer.

   (c) No change.

2. $\hat{\mu}$ is calculated by the following process:

   (a) Omit estimate of σ_1.

   (b) Replace σ_1 by σ.

   (c) Calculate only a, a', b, b', c and c'.

(d) Solve for $\Delta\mu$.

$$-(a - a') = \frac{b - c - b' - c'}{\sigma} \Delta\mu$$

(e) and (f) Omit reference to σ.

3. Precision of $\hat{\mu}$ is calculated as follows: (optional)

   (a) No change except that $\hat{\sigma} = \sigma$.

   (b) Calculate only the quantities A and 1/A.

4. The following results are typed out by the computer.

   (a) μ_0 and successive provisional solutions as well as values of $\Delta\mu$.

   (b) $\sigma_{\hat{\mu}} = 1/A$. (optional)

COMMENTS

The estimation procedure used in this program for statistical estimation of unknown parameters has a number of desirable properties (unbiased, minimum variance estimates) when the number of trials is large. Relatively little is known from a purely mathematical point of view about the estimates when the number of trials is small. Results of a number of investigations using simulated trials as well as experimental data indicate that the estimates of μ are unbiased for practical purposes but that estimates of σ on the average are too small. (See First Report on Ord Project TB4-005B, Rpt No. DPS-TB3-10/2 dtd July 1957 for a discussion of some of the results of various comparisons.) It is believed that estimates of μ can be used unhesitatingly even when samples are very small. Estimates of σ, however, are not only biased but are extremely variable under such conditions and must be used with utmost caution. Since the estimates of precision of $\hat{\mu}$ and $\hat{\sigma}$ are proportional to $\hat{\sigma}$, the statistical variability of the latter is reflected in them. Guidelines for the use of these statistics for very small samples have not as yet been developed. (For small samples the experimental technique, such as up-and-down, is probably more influential in determining these statistics than is the estimation procedure.)

An essential, preliminary step in the application of the procedure to small samples is a careful examination of the data. First, if no zone of mixed results exists, no estimate of σ (other than zero) can be obtained. Second, the probability of success is an increasing function of the stimulus x. Therefore, if the highest value of x resulted in a failure, or the lowest value of x resulted in a success, attempts to estimate σ will generally be futile. The process

usually diverges with σ increasing without limit. This situation is the result of incomplete testing (sometimes unavoidable) and can not be rectified at the data analysis stage. Third, if there is no ZMR and a value of σ is assumed in order to obtain an estimate of μ, the estimate of the precision of $\hat{\mu}$ should not be relied upon since it is proportional to the assumed σ. In other words, choice of a different value for σ would probably change $\hat{\mu}$ very little but would change its precision estimate.

Although the computer program was developed for the analysis of test data, it also provides a means of conducting further work, employing simulated experiments from known distributions to determine properties of this estimation procedure for small samples.

EXAMPLE

Two examples are given to illustrate the use of this program. The first is taken from a test to determine fuze arming distance and uses the program for grouped data. The second is from a test of a projectile for penetration of armor and is an example of ungrouped data having a zone of mixed results.

Example 1. The following data was obtained in the experiment.

Distance from Muzzle to Target, in.	No. of Rd Fired	No. of Fuzes Armed
575	9	1
581	10	3
587	10	3
593	10	5
602	10	6
606	8	6

These data are fed to the computer with the distances as x_i, the number of rounds fired as n_i, and the number of fuzes armed as m_i. A plot of ratios, m_i/n_i on normal probability paper suggests $\mu_o = 596$ and $\sigma_o = 18$. δ was chosen as 0.1. The computer types the following results. (The computer automatically prints 7 decimals in all results unless special instructions are inserted.)

i	μ_i	$\Delta_{i+1}\mu$	σ_i	$\Delta_{i+1}\sigma$
0	596.0000000	-1.2096147	18.0000000	.6334687
1	594.7903852	- .0100556	18.6334687	.3960649
2	594.7803295	.0016863	19.0295337	.0265457

$\hat{\mu} = 594.7820159$ $\hat{\sigma} = 19.0560795$

var $\hat{\mu}$ = 12.6976265
var $\hat{\sigma}$ = 39.1847815
cov $\hat{\mu},\hat{\sigma}$ = 7.2206360

The standard errors of $\hat{\mu}$ and $\hat{\sigma}$ are 3.56 and 6.26 respectively.

Often a confidence interval for a given percentile is desired. Suppose the 5th percentile is to be estimated as well as a confidence interval for it.

$$\hat{P}_5 = \hat{\mu} - 1.64\,\hat{\sigma} = 563.5$$

Since $\hat{\mu}$ and $\hat{\sigma}$ are jointly normally distributed with variances and covariance as calculated above, $\hat{P}_5$ is also normally distributed.

$$\text{Var } \hat{P}_5 = \text{var } \hat{\mu} + \text{var } \hat{\sigma}\,(1.64)^2 - 2 \quad (1.64) \text{ cov } \hat{\mu},\hat{\sigma}.$$

Substitution of calculated values gives var $\hat{P}_5 = 94.41$, and standard deviation of $\hat{P}_5 = 9.71$. Thus the interval can be written as

$$563.5 - K9.71 < P_5 < 563.5 + K9.71$$

where K is chosen from normal tables to give the desired confidence coefficient. For example, K = 1.96 for a coefficient of 95%, and

$$544.5 < P_5 < 582.5.$$

Example 2. The following data were obtained in a penetration test of a projectile.

Striking Velocity of Projectile, fps	Results
636	partial penetration
769	" "
828	complete "
844	" "
812	partial "
859	" "
893	complete "
859	" "

The data are transcribed for the computer as

x_i	δ_i
636	0
769	0
828	1
844	1
812	0
859	0
893	1
859	1

μ_0, σ_0 are estimated by the computer. d is taken as 0.1.

The computer types the following table

i	μ_i	$\Delta_{i+1}\mu$	σ_i	$\Delta_{i+1}\sigma$
0	841.5000000	-10.8213909	35.0000000	- .9584355
1	830.6786090	- .6187483	34.0415644	3.7175175
2	830.0598606	- .1775926	37.7590319	1.1489502
3	829.8822680	- .0131477	38.9080322	.0880689
4	829.8691202	- .0000580	38.9961012	.0004379

$\hat{\mu}$ = 829.8690622 $\quad\hat{\sigma}$ = 38.9965391

var $\hat{\mu}$ = 481.5176056
var $\hat{\sigma}$ = 800.6888854
cov $\hat{\mu},\hat{\sigma}$ = -147.0445795

1 Incl
v(t) functions

John S. Hagan
JOHN S. HAGAN
Actg Chief
Analytical Laboratory

Approximation of Certain Functions in BRL Technical Note 151

In maximum likelihood procedures for estimation of parameters mu and sigma of the normal curve from sensitivity data functions Z/P and Z/Q are needed. Since P and Q are definite integrals, their repeated calculation is time consuming. Consequently, for machine programming of the procedure approximations that reduce the calculations have been developed.

$$z(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

$$P(t) = \int_{-\infty}^{t} z(x)\, dx$$

$$v(t) = \frac{z(t)}{1 - P(t)} \qquad v(t) = Z/Q, \text{ and } v(-t) = Z/P \text{ in BRL Note 151}$$

t from	t to	
$-\infty$	-2.5	$v(t) = .398942\, e^{-t^2/2}$

$$v(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$$

		c_0	c_1	c_2	c_3
-2.5	-1.5	.8911463	.8428656	.2727600	.03014958
-1.5	-1.0	.8187139	.7014965	.1801986	.00981483
-1.0	- .3	.7985070	.6409503	.1189404	-.01111111
- .3	1.0	.7979054	.6365250	.1084799	-.01780600
1.0	2.5	.7847894	.6702693	.0780463	-.008037033
2.5	5.0	.6873630	.7831186	.03346049	-.002025425
5.0	∞	.18629	1.000000	0	0

Incl 1